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A Quantitative Korovkin Theorem for Random Functions with Multivariate Domains

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A quantitative Korovkin theorem for random functions with compact convex parameter spaces is established.

1. INTRODUCTION

Consider the problem of approximating a real-valued continuous function defined on a compact interval $[a, b] \subset \mathbb{R}$. The efficiency of a given sequence f_n of approximations can be expressed by means of

$$\|f - f_n\| = \sup_{t \in [a,b]} |f(t) - f_n(t)|.$$

If $f_n = T_n f$ emerges from a monotone linear transformation T_n: $C[a, b] \to C[a, b]$, then a celebrated theorem of Korovkin states that the relation

$$\lim_{n \to \infty} \|f - T_n f\| = 0$$

holds for arbitrary $f \in C[a, b]$ provided this relation has been verified for the three test functions $f(t) = 1$, $f(t) = t$, and $f(t) = t^2$. Moreover, there exist quantitative versions of this theorem. In the literature, numerous generalizations and connections with related topics have been established. See, e.g., Berens and Lorentz [3], Censor [5], DeVore [6], Donner [7], Gonska [8], Mond and Vasudevan [10], Nishishiraho [11], Roth [12], Sablonnière [13], Scheffold [14], Schempp [15], and Wolff [18]. Results relevant in stochastics were given by Anastassiou [1, 2] and Hahn [9]. The quantitative results discussed so far usually deal with "sure" or "deterministic" mappings, i.e., the value $f(t)$ is uniquely determined by the argument t. However, any approximation problem concerning sure functions can be posed equally well for "random" functions, and many applications,

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such as stochastic simulation, exclusively depend on random functions. For example, consider the problem of constructing the random curve of a moving particle undergoing molecular bombardment in a liquid assuming a finite number of positions have been observed.

If a random function (or stochastic process) $X(t, \omega)$, $t \in [a, b]$, $\omega \in \Omega$, is to be approximated by monotone transformations T_n then the maximal error in the qth mean

$$\|X - T_n X\| = \sup_{t \in [a,b]} (E\,|X(t, \omega) - (T_n X)(t, \omega)|^q)^{1/q}$$
$$= \sup_{t \in [a,b]} \left(\int_\Omega |X(t, \omega) - (T_n X)(t, \omega)|^q \, P(d\omega) \right)^{1/q}$$

with respect to the underlying probability space $(\Omega, \mathscr{A}, P)$ shows the effectiveness of T_n.

In order to prove stochastic analogues of the above Korovkin theorem it turns out that monotonicity alone is not sufficient. Under mild additional assumptions, a qualitative theorem can be found in [16]; for processes with parameter sets $K = [a, b]$ a quantitative version for $q = 2$ is given in [17] which makes use of the classical Korovkin test family. It is the purpose of this paper to discuss the multivariate case: $X(t, \omega)$ is a stochastic process with a convex compact parameter set K, $q \geqslant 1$ is arbitrary, and the test family is expressed in terms of a given set of continuous linear functionals.

2. Basic Definitions

$(\Omega, \mathscr{A}, P)$ denotes a fixed probability space and $L^q(\Omega, \mathscr{A}, P)$, $q \geqslant 1$, is the set of all real-valued random variables with finite qth moments, i.e., the set of all $(\Omega, \mathscr{A}) - (\mathbb{R}, \mathscr{B})$-measurable mappings $Z = Z(\omega)$ with

$$\|Z(\omega)\|_q = (E\,|Z(\omega)|^q)^{1/q} = \left(\int_\Omega |Z(\omega)|^q \, P(d\omega) \right)^{1/q} < \infty,$$

where $\mathscr{B}$ is the σ-field of Borel subsets of $\mathbb{R}$. K stands for a compact convex subset of a real normed vector space V, and $\Gamma = \{\gamma_1, ..., \gamma_m\}$, $m \geqslant 1$, is a fixed set of bounded linear functionals on V. Γ is assumed to be separating: given $v, w \in V$ with $v \neq w$ one can find a functional $\gamma \in \Gamma$ satisfying $\gamma(v) \neq \gamma(w)$.

$X = X(t, \omega)$ always denotes a stochastic process with index set K and real state space $(\mathbb{R}, \mathscr{B})$. Sums, etc., of stochastic processes are to be understood pointwise, and the natural ordering in $\mathbb{R}$ induces the canonical ordering

$$X \leqslant Y \qquad \text{iff} \qquad X(t, \omega) \leqslant Y(t, \omega) \text{ for each } t \in K.$$

(Passing to equivalence classes, processes which coincide with probability one for each fixed $t \in K$ will not be distinguished between.) The vector lattice of processes with bounded qth moments is defined to be

$$B_\Omega(K) = \{X: \sup_{t \in K} \|X(t, \omega)\|_q < \infty\}$$

and becomes a normed vector lattice by means of the norm

$$\|X\| = \sup_{t \in K} \|X(t, \omega)\|_q.$$

The space

$$C_\Omega(K) = C(K, L^q(\Omega, \mathscr{A}, P))$$

of L^q-continuous processes is a linear sublattice of $B_\Omega(K)$, and the corresponding spaces

$$B(K) = \{f: \sup_{t \in K} |f(t)| < \infty\}$$

and

$$C(K) = C(K, \mathbb{R})$$

of "ordinary" real-valued functions can be embedded if we identify a function $f(t)$ with the degenerate process $X_f(t, \omega) = f(t)$, $t \in K$, $\omega \in \Omega$. The nonrandom theory about quantitative Korovkin theorems is obtained if a trivial probability space is considered which contains a single element only. (Then

$$B_\Omega(K) = B(K), \qquad C_\Omega(K) = C(K)$$

holds true.)

Given a process $X \in B_\Omega(K)$, the mean value function EX and the norm function $X_{||}$ are defined according to

$$(EX)(t) = E(X(t, \omega)) = \int_\Omega X(t, \omega)\, P(d\omega)$$

and

$$X_{||}(t) = \|X(t, \omega)\|_q = \left(\int_\Omega |X(t, \omega)|^q\, P(d\omega)\right)^{1/q}.$$

Smoothness of a process $X \in B_\Omega(K)$ with respect to the system Γ is expressed by the stochastic modulus of continuity

$$\eta(X; \Gamma; \delta) = \sup \left\{ \|X(t_1, \omega) - X(t_2, \omega)\|_q : t_1, t_2 \in K, \sum_{i=1}^{m} (\gamma_i(t_1) - \gamma_i(t_2))^2 \leqslant \delta^2 \right\},$$

where δ is nonnegative.

Some properties of processes $X \in B_\Omega(K)$ are summarized in the following lemma.

LEMMA 2.1. *Assume* $X \in B_\Omega(K)$.

(i) *The mean value function* EX *and the norm function* $X_{||}$ *are elements of* $B(K)$.

(ii) *If* X *lies in* $C_\Omega(K)$ *then* EX *and* $X_{||}$ *are elements of* $C(K)$.

(iii) *The stochastic modulus of continuity satisfies*

$$\eta(X; \Gamma; \alpha \cdot \delta) \leqslant (1 + \alpha) \cdot \eta(X; \Gamma; \delta)$$

for each $\alpha > 0$ *and* $\delta \geqslant 0$.

(iv) $X \in C_\Omega(K)$ *implies* $\lim_{\delta \to 0^+} \eta(X; \Gamma; \delta) = 0$.

3. MONOTONE OPERATORS FROM $C_\Omega(K)$ INTO $B_\Omega(K)$

DEFINITION 3.1. A bounded linear mapping $T: C_\Omega(K) \to B_\Omega(K)$ is called monotone if $X \geqslant 0$ implies $TX \geqslant 0$ for each $X \in C_\Omega(K)$.

As mentioned before, monotonicity alone does not guarantee stochastic analogues of quantitative Korovkin theorems.

First of all, it is required that application of the operators T and E can be interchanged.

DEFINITION 3.2. A mapping $T: C_\Omega(K) \to B_\Omega(K)$ is named to be E-commutative provided

$$E(TX) = T(EX)$$

is satisfied for each $X \in C_\Omega(K)$.

A second condition states that images of the most "simple" random functions are not "too complicated". The most simple random functions can be described as follows: if $A \in \mathscr{A}$ is a given event and if $\omega \in A$ occurs (with probability $P(A)$), then the random function—being independent from $t \in K$—is equal to one; if $\omega \notin A$ occurs (with probability $1 - P(A)$) then the random function vanishes identically. Hence for each event $A \in \mathscr{A}$ the "simple" process S_A is defined to be

$$S_A(t, \omega) = 1_A(\omega) = \begin{cases} 1, & \text{if } \omega \in A \\ 0, & \text{if } \omega \notin A. \end{cases}$$

Stochastic simplicity of an operator T states that S_A is always deformed by means of a real-valued function a_T.

DEFINITION 3.3. A mapping $T: C_\Omega(K) \to B_\Omega(K)$ is called stochastically simple if there exists a real-valued function $a_T: K \to \mathbb{R}$ such that

$$(TS_A)(t, \omega) = a_T(t) \cdot S_A(t, \omega)$$

holds true for each event $A \in \mathscr{A}$.

Note that in the nonrandom case any mapping is E-commutative and any mapping leaving the vanishing function $S_\varnothing \equiv 0$ invariant is stochastically simple.

The additional conditions formulated in the above definitions are rather mild and may be verified immediately for operators which are defined by summation or integration, such as

$$(TX)(t, \omega) = \sum_{k=1}^{n} D_k(t) \cdot X(t_k, \omega),$$

where D_k are given functions on K and $t_k \in K$ are fixed knots. See also [16].

The mean value operator EX is *not* stochastically simple.

LEMMA 3.4. (i) *If T is E-commutative then T maps the subspace $C(K)$ into $B(K)$.*

(ii) *Assume T to be monotone, E-commutative, and stochastically simple. If $Z(\omega)$ is a random variable with $\|Z(\omega)\|_q < \infty$ and $f \in C(K)$ a continuous function, then the image of the process*

$$f_Z(t, \omega) = f(t) \cdot Z(\omega)$$

is given by

$$(Tf_Z)(t, \omega) = (Tf)(t) \cdot Z(\omega).$$

Proof. (i) $f \in C(K)$ implies $Tf = T(Ef) = E(Tf) \in B(K)$.

(ii) If $Z(\omega) = 1_A(\omega)$, $A \in \mathscr{A}$, is an indicator the relation

$$-\|f\| \cdot 1_A(\omega) \leqslant f(t) \cdot 1_A(\omega) \leqslant \|f\| \cdot 1_A(\omega)$$

gives

$$-\|f\| \cdot a_T(t) \cdot 1_A(\omega) \leqslant (Tf_{1_A})(t, \omega) \leqslant \|f\| \cdot a_T(t) \cdot 1_A(\omega).$$

Therefore, $(Tf_{1_A})(t, \omega) = 0$ if $\omega \notin A$. Since A was arbitrary, the equation

$$(Tf)(t) = (Tf_{1_A})(t, \omega) + (Tf_{1_{\Omega - A}})(t, \omega)$$

implies

$$(Tf_{1_A})(t, \omega) = (Tf)(t) \cdot 1_A(\omega).$$

(iii) The assertion for arbitrary random variables $Z(\omega)$ follows from (ii) by the choice of a sequence $Z_n(\omega)$ of primitive variables with $\lim_{n \to \infty} \|Z(\omega) - Z_n(\omega)\|_q = 0$. ∎

A main tool in proving quantitative Korovkin theorems is the inequality $|Tf| \leqslant T|f|$ for $f \in C(K)$. Being trivial in the nonrandom case, the stochastic analogue $(TX)_{||} \leqslant T(X_{||})$ cannot be verified directly since we have to deal with the L^q-norm rather than with the absolute value induced by the given ordering. First, an auxiliary result is required for integrals on compact spaces K. (Convexity is not needed here.)

LEMMA 3.5. *Let $(K, \mathscr{K}, \mu)$ be a measure space where K is a compact topological Hausdorff space, $\mathscr{K}$ the σ-field of Borel subsets of K, and μ a finite measure. Then there is a bounded linear operator*

$$\Phi\colon C_\Omega(K) \to L^q(\Omega, \mathscr{A}, P)$$

with

$$\Phi(f_{1_A}) = 1_A(\omega) \cdot \int_K f(t)\, \mu(dt)$$

for each continuous function $f \in C(K)$ and each event $A \in \mathscr{A}$. Moreover, Φ is unique and satisfies

$$\|\Phi(X)\|_q \leqslant \int_K X_{||}(t)\, \mu(dt)$$

for each process $X \in C_\Omega(K)$. *In particular,*

$$\|\Phi\| = \mu(K).$$

Proof. (i) Given $X \in C_\Omega(K)$ there exist random variables $Z_k(\omega)$, $\|Z_k(\omega)\|_q < \infty$, and functions $f_k(t) \in C(K)$, $1 \leqslant k \leqslant n$, such that

$$\left\| X(t, \omega) - \sum_{k=1}^{n} Z_k(\omega) \cdot f_k(t) \right\| < \varepsilon$$

for any given $\varepsilon > 0$. (See Bourbaki [4].) Passing to indicators one finds that the linear subspace

$$D_\Omega(K) = \left\{ Y \in C_\Omega(K) \colon Y(t, \omega) = \sum_{k=1}^{n} 1_{A_k}(\omega) \cdot f_k(t), f_k \in C(K), \right.$$
$$\left. A_k \in \mathscr{A}, A_k \cap A_l = \varnothing \text{ for } k \neq l, \sum_{k=1}^{n} A_k = \Omega \text{ for some } n \in \mathbb{N} \right\}$$

is dense in $C_\Omega(K)$.

(ii) If Y lies in $D_\Omega(K)$ having the above representation set then

$$\Phi(Y) = \sum_{k=1}^{n} 1_{A_k}(\omega) \cdot \int_K f_k(t)\, \mu(dt).$$

It may be checked that Φ is a well defined linear mapping from $D_\Omega(K)$ into $L^q(\Omega, \mathscr{A}, P)$ with

$$\Phi(f_{1_A}) = \Phi(f(t) \cdot 1_A(\omega)) = 1_A(\omega) \cdot \int_K f(t)\, \mu(dt).$$

(iii) If $g_1, ..., g_I \in C(K)$ are fixed functions, $I \in \mathbb{N}$, then for each $\varepsilon > 0$ there exist points $t_1, ..., t_M \in K$ and subsets $K_1, ..., K_M \subset K$, $M \in \mathbb{N}$, with the following properties: the sets K_m are pairwise disjoint elements of $\mathscr{K}$ with $\sum_{m=1}^{M} K_m = K$ satisfying

$$\left| \int_K g_i(t)\, \mu(dt) - \sum_{m=1}^{M} g_i(t_m) \cdot \mu(K_m) \right| \leqslant \varepsilon$$

for each $i = 1, ..., I$. (For if $\varepsilon > 0$ is given then—due to the compactness of K—one can find points $t_1, ..., t_M \in K$ and open sets $O_1, ..., O_M$ such that K is covered by $\bigcup_{m=1}^{M} O_m$, O_m contains t_m, $1 \leqslant m \leqslant M$, and

$$|g_i(t) - g_i(t_m)| \leqslant \frac{\varepsilon}{\mu(K)}$$

for each $t \in O_m$ and $i = 1, ..., I$. Putting $K_1 = O_1$ and $K_m = O_m - \bigcup_{j=1}^{m-1} O_j$ for $m \geq 2$ we get

$$\left| \int_K g_i(t)\, \mu(dt) - \sum_{m=1}^{M} g_i(t_m)\, \mu(K_m) \right|$$

$$\leq \sum_{m=1}^{M} \int_{K_m} |g_i(t) - g_i(t_m)|\, \mu(dt) \leq \varepsilon.)$$

(iv) Hold an element $Y \in D_\Omega(K)$ fixed with

$$Y(t, \omega) = \sum_{k=1}^{n} 1_{A_k}(\omega) \cdot f_k(t).$$

Suppose an arbitrary $\varepsilon > 0$ is given.

By Lemma 2.1, $Y_{||}$ is continuous on K. Applying the assertion of part (iii) on the functions $f_1, ..., f_n$ and $Y_{||}$ we obtain

$$\|\Phi(Y)\|_q = \left\| \sum_{k=1}^{n} 1_{A_k}(\omega) \cdot \int_K f_k(t)\, \mu(dt) \right\|_q$$

$$\leq \left\| \sum_{k=1}^{n} 1_{A_k}(\omega) \cdot \sum_{m=1}^{M} f_k(t_m)\, \mu(K_m) \right\|_q + \varepsilon$$

$$\leq \sum_{m=1}^{M} \left\| \sum_{k=1}^{n} 1_{A_k}(\omega) \cdot f_k(t_m) \right\|_q \cdot \mu(K_m) + \varepsilon$$

$$= \sum_{m=1}^{M} \|Y(t_m, \omega)\|_q \cdot \mu(K_m) + \varepsilon$$

$$\leq \int_K Y_{||}(t)\, \mu(dt) + 2\varepsilon.$$

This implies the inequality

$$\|\Phi(Y)\|_q \leq \int_K Y_{||}(t)\, \mu(dt) \leq \mu(K) \cdot \|Y\|.$$

Hence Φ is bounded with norm $\|\Phi\| = \mu(K)$ since the constant function $\gamma_0(t) \equiv 1$ yields

$$\|\Phi(\gamma_0)\|_q = \left| \int_K \gamma_0(t)\, \mu(dt) \right| = \mu(K) \cdot \|\gamma_0\|.$$

(v) On condition that an element $X \in C_\Omega(K)$ is given, choose a sequence $Y_n \in D_\Omega(K)$ with $\lim_{n \to \infty} \|X - Y_n\| = 0$ and set

$$\Phi(X) := \lim_{n \to \infty} \Phi(Y_n),$$

where the limit is to be understood in the L^q-sense. Then Φ is the desired operator on $C_\Omega(K)$, and Φ is uniquely determined by the condition

$$\Phi(f_{1_A}) = 1_A(\omega) \cdot \int_K f(t)\, \mu(dt)$$

because $D_\Omega(K)$ is dense in $C_\Omega(K)$. ∎

Formally, $\Phi(X)$ may be viewed as a kind of stochastic L^q-integral; in the special case $K = [a, b] \subset \mathbb{R}$ and $q = 2$ the operator $\Phi(X)$ coincides with the ordinary stochastic Riemann integral.

LEMMA 3.6. *Let $T: C_\Omega(K) \to B_\Omega(K)$ be monotone, E-commutative, and stochastically simple. Then*

$$(TX)_{||} \leqslant T(X_{||})$$

is valid for each $X \in C_\Omega(K)$.

Proof. Hold $t \in K$ fixed. Define the real functional φ_t on $C(K)$ according to

$$\varphi_t(f) = (Tf)(t).$$

φ_t is bounded, linear, and monotone. By the Riesz representation theorem one can conclude

$$\varphi_t(f) = \int_K f(s)\, \mu_t(ds)$$

for each $f \in C(K)$, where μ_t is a finite measure defined on the σ-field $\mathscr{K}$ of the Borel subsets of K.

If we set

$$\Phi_t(X) = (TX)(t, \omega)$$

then Φ_t is a bounded linear operator from $C_\Omega(K)$ into $L^q(\Omega, \mathscr{A}, P)$. By Lemma 3.4, Φ_t satisfies

$$\Phi_t(f_{1_A}) = 1_A(\omega) \cdot (Tf)(t) = 1_A(\omega) \cdot \int_K f(s)\, \mu_t(ds).$$

Therefore, application of Lemma 3.5 yields

$$\begin{aligned}(TX)_{||}(t) &= \|(TX)(t, \omega)\|_q \\ &= \|\Phi_t(X)\|_q \leqslant \int_K X_{||}(s)\, \mu_t(ds) \\ &= \varphi_t(X_{||}) = (T(X_{||}))(t). \quad \blacksquare\end{aligned}$$

Apart from Korovkin theory, Lemma 3.6 might also serve as a source of various inequalities for monotone operators by specialization of the underlying probability space $(\Omega, \mathscr{A}, P)$. For example, there is the following expressive interpretation: assume $f_1, ..., f_n$ to be continuous functions on the real interval $[a, b]$ and let

$$\mathbf{f}(t) := (f_1(t), ..., f_n(t)), \qquad t \in [a, b]$$

be a path describing the motion of a particle. Given n nonnegative numbers p_k with $\sum_{k=1}^{n} p_k = 1$, then

$$(\rho\mathbf{f})(t) = \left(\sum_{k=1}^{n} p_k \cdot |f_k(t)|^q\right)^{1/q}$$

describes the weighted distance of the particle from the origin with respect to $q \geqslant 1$ and the weights p_k. Particularly, $p_1 = p_2 = \cdots = p_n = n^{-1}$ and $q = 2$ yield the Euclidean distance (up to the constant term $n^{-1/2}$). Consider now a monotone deformation of the path due to

$$(T\mathbf{f})(t) := ((Tf_1)(t), ..., (Tf_n)(t)).$$

On condition that a discrete probability space $\Omega = \{\omega_1, ..., \omega_n\}$ with probabilities $P\{\omega_k\} = p_k$ is chosen, application of Lemma 3.6 on the process

$$X(t, \omega) = \sum_{k=1}^{n} 1_{\{\omega_k\}}(\omega) \cdot f_k(t)$$

gives

$$(\rho T\mathbf{f})(t) \leqslant (T\rho\mathbf{f})(t),$$

i.e., the distance of the particle tracing out the transformed curve is always bounded by the transformation of the original distance. Similarly, more complicated "distances" can be treated if more complicated probability spaces are discussed.

4. The Rate of Convergence

Theorem 4.1 establishes an upper bound for the approximation error $\|X-TX\|$. Being bounded linear functionals on the space $V \supset K$, the elements of the system $\Gamma=\{\gamma_1, ..., \gamma_m\}$ may also be viewed as elements of $C(K)$. In the sequel the following abbreviations are used:

$$\gamma_0(t)=1, \qquad t \in K$$

and

$$\gamma_{i,t_0}(t)=(\gamma_i(t)-\gamma_i(t_0))^2, \qquad t \in K$$

for each $t_0 \in K$ and $1 \leqslant i \leqslant m$.

Theorem 4.1. *Assume that $T: C_\Omega(K) \to B_\Omega(K)$ is monotone, E-commutative, and stochastically simple. Given $\delta>0$, each process $X \in C_\Omega(K)$ satisfies*

$$\|X-TX\| \leqslant \|X\| \cdot \|\gamma_0 - T\gamma_0\| + \eta(X;\Gamma;\delta) \cdot \left(\|T\gamma_0\| + \delta^{-2} \cdot \sup_{t \in K} \sum_{i=1}^{m} T\gamma_{i,t}(t) \right).$$

Proof. Hold $t_0 \in K$ fixed. Any $t \in K$ with $\sum_{i=1}^{m} (\gamma_i(t)-\gamma_i(t_0))^2 > \delta^2$ fulfills

$$\begin{aligned} &\|X(t,\omega)-X(t_0,\omega)\|_q \\ &\quad \leqslant \eta(X;\Gamma;\delta) \cdot \left(1+\delta^{-1} \cdot \left(\sum_{i=1}^{m} (\gamma_i(t)-\gamma_i(t_0))^2\right)^{1/2}\right) \\ &\quad \leqslant \eta(X;\Gamma;\delta) \cdot \left(1+\delta^{-2} \cdot \sum_{i=1}^{m} (\gamma_i(t)-\gamma_i(t_0))^2\right). \end{aligned}$$

Since the last bound for $\|X(t,\omega)-X(t_0,\omega)\|_q$ applies equally well for $t \in K$ with $\sum_{i=1}^{m} (\gamma_i(t)-\gamma_i(t_0))^2 \leqslant \delta^2$ we can conclude that the process

$$Y_{t_0}(t,\omega)=X(t,\omega)-X(t_0,\omega) \cdot \gamma_0(t)$$

satisfies

$$(Y_{t_0})_{||} \leqslant \eta(X;\Gamma;\delta) \cdot \left(\gamma_0+\delta^{-2} \cdot \sum_{i=1}^{m} \gamma_{i,t_0}\right).$$

Because T is monotone, we obtain

$$T((Y_{t_0})_{||}) \leqslant \eta(X;\Gamma;\delta) \cdot \left(T\gamma_0+\delta^{-2} \cdot \sum_{i=1}^{m} T\gamma_{i,t_0}\right).$$

By Lemma 3.6,

$$(TY_{t_0})_{||} \leqslant \eta(X; \Gamma; \delta) \cdot \left(T\gamma_0 + \delta^{-2} \cdot \sum_{i=1}^{m} T\gamma_{i,t_0} \right).$$

This is an inequality between two functions, i.e., the inequality holds for arbitrary arguments. If we choose the argument $t = t_0$, then Lemma 3.4 and the definition of $Y_{t_0}(t, \omega)$ yield

$$\|(TX)(t_0, \omega) - X(t_0, \omega) \cdot (T\gamma_0)(t_0)\|_q$$
$$\leqslant \eta(X; \Gamma; \delta) \cdot \left((T\gamma_0)(t_0) + \delta^{-2} \cdot \sum_{i=1}^{m} (T\gamma_{i,t_0})(t_0) \right).$$

t_0 was arbitrary; consequently, taking the sup-norm on both sides implies

$$\|TX - X \cdot T\gamma_0\| \leqslant \eta(X; \Gamma; \delta) \cdot \left(\|T\gamma_0\| + \delta^{-2} \cdot \sup_{t \in K} \sum_{i=1}^{m} T\gamma_{i,t}(t) \right).$$

By Lemma 3.4, $T\gamma_0 - \gamma_0$ lies in $B(K)$. Thus

$$\|X \cdot T\gamma_0 - X\| \leqslant \|X\| \cdot \|T\gamma_0 - \gamma_0\|$$

and the assertion follows from

$$\|TX - X\| \leqslant \|X \cdot T\gamma_0 - X\| + \|TX - X \cdot T\gamma_0\|. \quad \blacksquare$$

COROLLARY 4.2. *Let T_n be a sequence of monotone, E-commutative, and stochastically simple operators. If δ_n is a sequence of positive real numbers with*

$$\sup_{t \in K} \sum_{i=1}^{m} T_n \gamma_{i,t}(t) \leqslant \beta \cdot \delta_n^2$$

for some constant $\beta \geqslant 0$ then

$$\|X - T_n X\| \leqslant \|X\| \cdot \|\gamma_0 - T_n \gamma_0\| + (\|T_n \gamma_0\| + \beta) \cdot \eta(X; \Gamma; \delta_n)$$

is valid for each process $X \in C_\Omega(K)$. Particularly, the conditions

$$\lim_{n \to \infty} \|\gamma_i - T_n \gamma_i\| = 0, \qquad 0 \leqslant i \leqslant m,$$

and

$$\lim_{n \to \infty} \|\gamma_i^2 - T_n(\gamma_i^2)\| = 0, \qquad 1 \leqslant i \leqslant m,$$

already imply

$$\lim_{n \to \infty} \|X - T_n X\| = 0$$

for each process $X \in C_\Omega(K)$.

Given a sequence of monotone operators being used to approximate a stochastic process X we must therefore check for E-commutativity and stochastic simplicity; usually, this would not cause difficulties since many operators are defined by means of summation or integration, and such operators are automatically E-commutative and stochastically simple.

Finally, we have to analyze the stochastic modulus of continuity $\eta(X; \Gamma; \delta)$ in order to describe the asymptotic behaviour. By definition of $\eta(X; \Gamma; \delta)$ this can be done by considering the covariance structure of X with respect to q. For example, assume $q = 2$, let K be a subset of the m-dimensional Euclidean space, and consider the canonical system $\Gamma = \{\gamma_1, ..., \gamma_m\}$ which consists of the projections

$$\gamma_i(t) = \tau_i, \qquad t = (\tau_1, ..., \tau_m) \in \mathbb{R}^m.$$

If $\|\cdot\|_E$ stands for the Euclidean norm then $\eta(X; \Gamma; \delta)$ becomes

$$\eta(X; \Gamma; \delta) = \sup_{\substack{t_1, t_2 \in K \\ \|t_1 - t_2\|_E \leqslant \delta}} (f_X(t_1, t_2))^{1/2},$$

where $f_X(t_1, t_2)$ can be expressed in terms of the mean value function $(EX)(t)$, the function of variances $(\mathrm{Var}\, X)(t)$, and the covariance function $(\mathrm{Cov}\, X)(t_1, t_2)$ according to

$$\begin{aligned} f_X(t_1, t_2) = ((EX)(t_1) - (EX)(t_2))^2 + (\mathrm{Var}\, X)(t_1) \\ + (\mathrm{Var}\, X)(t_2) - 2(\mathrm{Cov}\, X)(t_1, t_2). \end{aligned}$$

Remark. The assertions of Theorem 4.1 and Corollary 4.2 remain valid mutatis mutandis if the space $L^q(\Omega, \mathscr{A}, P)$ and the norm $\|\cdot\|_q$ are replaced by a vector sublattice $L \subset \mathbb{R}^\Omega$ and a lattice semi-norm ρ on L, where L contains the constant function $Z_0(\omega) \equiv 1$, $\omega \in \Omega$, and ρ satisfies $\rho(Z_0) > 0$. Since the mean value function may be undefined, and indicators may not be included in L, the hypothesis that T is E-commutative and stochastically simple must be interpreted as:

T maps $C(K)$ into $B(K)$ and satisfies

$T(f \cdot Z) = (Tf) \cdot Z$ for each $Z \in L$ and $f \in C(K)$.

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